

## EXISTENCE AND OPTIMAL CONTROL ANALYSIS OF HIV/AIDS MODEL

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### Abstract

*Human Immunodeficiency Virus/ Acquired Immune Deficiency Syndrome (HIV/AIDS) stands as a paramount global health concern, being both a prevalent sexually transmitted disease and one of the most catastrophic epidemics ever recorded, with profound impacts on human health. This study revisits the theme of Odebiyi et al. The study presents a mathematical model describing HIV/AIDS transmission dynamics, incorporating optimal control strategies for public sensitization and the use of antiretroviral treatment thereby providing new insight to their work. Analytical results via optimal control theory confirm the existence of optimal solutions. Numerical Simulation results illustrate the efficacy of public sensitization and treatment in controlling HIV/AIDS. The combined implementation of these control measures yields significant reductions in HIV transmission. The analysis underscores the importance of multi-faceted HIV control strategies. Our findings are expected to inform policymakers in developing targeted interventions, optimizing resource allocation, and enhancing the impact of HIV control programs.*

**Keywords:** Existence, HIV/AIDS, Optimal Control, Sensitization, Treatment

## 1. INTRODUCTION

Human Immunodeficiency Virus (HIV) is a relentless and destructive virus that wreaks havoc on individuals, communities, and economies worldwide. If left unchecked, HIV infection can spiral into a catastrophic crisis, ravaging lives and livelihoods. It had a significant impact on the economic globally such that economic growth and development are stifled.

Over time, optimal control models have played a crucial role in developing successful tactics to minimize the transmission of HIV among the population (Akudibillah et al., 2018; Gurmu et al., 2020; Olaniyi et al., 2020). It is essential to have efficient control methods in place to lessen the consequences of epidemics like HIV/AIDS and enhance the well-being of individuals affected by the disease (Zhao et al., 2022). Antiretroviral therapy (ART) are drugs that target the HIV life cycle with the aim of halting HIV replication and restoring immune function, thus slowing the progression to AIDS (Bräu et al., 2006; van Sighem et al., 2003). The expansion of antiretroviral treatment in low and middle income countries presents a unique chance to increase access to HIV treatments and improve prevention strategies against the HIV epidemic. Antiretroviral treatment has dramatically improved HIV/AIDS outcomes, allowing previously debilitated patients to regain health, achieve viral suppression, and lead productive lives (UNAIDS, 2002). Moreover, implementing the most effective

treatment plan results in a significant decrease in the occurrence and spread of the disease (Akudibillah et al., 2018).

Detecting and intervening early can greatly enhance health results and prevent fatalities. The availability of antiretroviral drugs has allowed numerous individuals suffering from advanced stages of HIV and AIDS to improve their health and experience a substantial decrease in viral load, sometimes reaching undetectable levels, thus enabling them to resume their regular daily activities (UNAIDS, 2002). More so, the rise in screening rates, along with a decrease in the transition from infection to the AIDS stage, is causing a reduction in the transmission of HIV/AIDS (Fauziah et al., 2024; Odebiyi et al., 2024; Okyere et al., 2020; Shabani et al., 2011). The act of identifying those who are contagious has been shown to have a notable impact on how the illness spreads. Counseling, guidance, and antiretroviral treatment have been identified as key tools in containing the spread of HIV/AIDS and have the potential to prevent approximately 9-12 million new cases within a 40-year timeframe (Ibrahim et al., 2021).

Since a cure for HIV/AIDS remains elusive, it is imperative to investigate and employ comprehensive strategies to prevent transmission, mitigate disease spread and reduce prevalence, ultimately curtailing the pandemic's impact (Iddi et al., 2012). Despite comprehensive HIV prevention and treatment efforts, countries struggle to meet epidemic control targets (Chazuka et al., 2024). Urgent action is required to expand testing and treatment, strengthen prevention programs, and support affected communities. Optimal control strategies are necessary to combat HIV transmission and inform policy decisions (Akudibillah et al., 2018).

A wide range of mathematical models, including those incorporating optimal control strategies, have been developed to understand and analyze HIV/AIDS transmission dynamics as evidenced by numerous researchers (Adepoju et al., 2024; Chazuka et al., 2024; Gurmu et al., 2020; Okosun et al., 2013; Okyere et al., 2020; Shabani et al., 2011; Sule & Abdullah, 2014). Optimal control models have been instrumental in guiding effective HIV prevention and control strategies. These models empower public health experts and policymakers with data-driven insights to identify high-impact interventions, optimize resource allocation, inform evidence-based decision making and maximize impact on HIV transmission reduction (Akudibillah et al., 2018; Gurmu et al., 2020; Okosun et al., 2013; Rabiou et al., 2021; Silva & Torres, 2017).

We designed a unique optimal control model in which we included two different types of controls: public sensitization such as mass media campaigns, community-based initiatives, educational programs, interpersonal communication, public events, digital platforms, partnerships and collaborations and also, the use of treatment with antiretroviral, as controls. The main goal of optimal control methods in HIV/AIDS modeling is to create strategies grounded in evidence.

The aim of this study is to establish the optimal combination of sensitization strategies and treatment with antiretroviral therapy (ART) that can effectively control HIV/AIDS transmission and improve health outcomes.

The rest of the project is structured as follows: In Section 2, the model formulation and its description are discussed, and in Section 3, the analysis of optimal control, proof of optimality, and characterization of optimal control are covered.

Section 4 delves into the numerical simulations for the optimal control analysis model, while Section 5 wraps up the work.

## 2. FORMULATION OF THE MODEL AND ITS DESCRIPTION

This present paper relies on the theoretical model proposed by Odebiyi et al. (2024) as its central framework. The model incorporates four classes to understand HIV infection dynamics disease progression, and the impact of interventions which includes public sensitization and ART treatment on the epidemic's trajectory.

In order to capture the dynamics of HIV, we propose a compartmental model consisting of four distinct classes representing different infection stages and conditions at time(t). The classes are:

- 1) The susceptible  $S(t)$  population consists of population members who have no immunity, whether natural or acquired, and are therefore at risk of contracting the disease.
- 2) Asymptomatic population  $I_1(t)$  are individuals who are infected with a disease but show no apparent symptoms or clinical signs.
- 3) Symptomatic infected  $I_2(t)$  consists of people infected with a disease who exhibit apparent and observable symptoms or clinical signs of illness.
- 4) AIDS  $A(t)$  represents the final stage of HIV infection, where the immune system is severely compromised, leaving individuals susceptible to numerous opportunistic infections such as Pneumonia, Tuberculosis, cancer, wasting syndrome and other related complications.

Timely ART initiation in all the classes is vital for optimal treatment, maximizes viral suppression and reduces disease progression. As a result, the said model considers a sexually active population of size  $N(t)$  at time  $t$ , given by  $N(t) = S(t) + I_1(t) + I_2(t) + A(t)$ .

The transfer diagram of four stages HIV/AIDS infection model are shown below in figure 1:

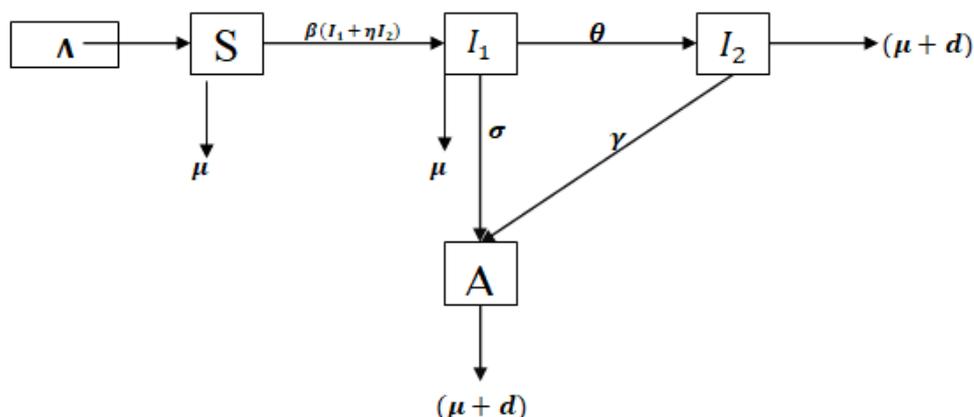


Figure 1. The model Schematic Diagram of an SI Model (Odebiyi et al., 2024)

### 2.1. Model Description

Recruitment of susceptible individuals ( $\pi$ ) happens when new individuals are born or move into the population. The Asymptomatic population  $I_1$  and Symptomatic population  $I_2$  transmits the infection to Susceptible individuals at distinct rates  $\beta$ , and  $\eta\beta$  respectively. It is assumed that  $\beta < \eta\beta$ , where  $0 < \eta < 1$ . Asymptomatic population can Screening of individuals without symptoms can be done at a certain rate  $\theta$ , leading them to develop symptoms. Additionally, the progression from being asymptomatic to having full-blown AIDS occurs at different rates  $\delta_1$  and  $\delta_2$  respectively for asymptomatic and symptomatic population. Some died naturally ( $\mu$ ) and some died due to AIDS related disease ( $d$ ).

The model's dynamics are determined by the interactions outlined in the following set of nonlinear ordinary differential equations, which elucidate the intricate connections between the elements.

$$\begin{aligned} \frac{dS}{dt} &= \pi - \beta(I_1 + \eta I_2)S - \mu S \\ \frac{dI_1}{dt} &= \beta(I_1 + \eta I_2)S - (\theta + \mu + \delta)I_1 \\ \frac{dI_2}{dt} &= \theta I_1 - (\mu + \delta_1 + d)I_2 \\ \frac{dA}{dt} &= \delta I_1 + \delta_1 I_2 - (\mu + d)A \\ S(0) &= S_0, I_1(0) = I_{1_0}, I_2(0) = I_{2_0}, A(0) = A_0 \end{aligned}$$

The model utilizes the defined model parameters as listed below.

**Table 1. Model Variables and Parameters**

Parameters and variables	Description
$\pi$	Recruitment rate of Susceptible
$\beta$	Transmission rate
$d$	AIDS related death rate
$\theta$	Pace at which asymptomatic population become aware of being infected after a screening process.
$\mu$	Natural mortality rate unrelated to AIDS
$d$	AIDS related death rate.
$\delta$	Progression rate from Asymptomatic class to AID class.
$\delta_1$	Progression rate from Symptomatic class to AID class.
$\eta$	Infectivity rate of transmission
$S(t)$	Susceptible population at a given time t.
$I_1(t)$	Asymptomatic population at a given time t.
$I_2(t)$	Symptomatic population at a given time t.
$A(t)$	AIDS population.

The model above has the disease-free equilibrium points as  $E^0 = (S^0, I_1^0, I_2^0, A^0) = \left(\frac{\pi}{\mu}, 0, 0, 0\right)$  which indicate that the disease has been completely eradicated. More so, we analyze the equations mentioned above by utilizing the matrix of the upcoming generation and calculated the fundamental reproduction number ( $R_0$ ) indicating the total number of new infections caused by an average person who is carrying the disease during the entire period of their contagiousness (Odebiyi et al., 2024). Therefore, the fundamental reproductive rate is calculated as  $R_0 = \frac{\beta\pi(\eta\theta + \mu + \delta_1 + d)}{\mu(\theta + \mu + \delta)(\mu + \delta_1 + d)}$ . The absence of disease is considered stable at a local level when  $R_0 < 1$ , suggesting that the illness will diminish and the population will go back to being free of the disease, and unstable if otherwise meaning that the disease will spread and will move away from the disease-free state.

### 3. ANALYSIS OF OPTIMAL CONTROL

The model for HIV/AIDS (1) has been enhanced to incorporate two optimal control mechanisms  $u_i(t), i = 1, 2$  where  $u_1(t)$  represents the control of public sensitization while  $u_2(t)$  represents the control of use of Antiretroviral treatment (ART). The resulting non-autonomous HIV/AIDS Model is provided by

$$\left. \begin{aligned} \frac{dS}{dt} &= \pi - (1 - u_1)\beta S(I_1 + \eta I_2) - \mu S \\ \frac{dI_1}{dt} &= (1 - u_1)\beta S(I_1 + \eta I_2) - (\theta + \delta + \mu)I_1 \\ \frac{dI_2}{dt} &= \theta I_1 - (\mu + u_2\delta_1)I_2 \\ \frac{dA}{dt} &= \delta I_1 + u_2\delta_1 I_2 - (\mu + d)A \end{aligned} \right\} \quad (1)$$

The non-autonomous system's (1) optimal control model is examined using Pontryagin's maximum principle [24] in order to reduce the number of infected individuals within the human population. The non-autonomous system's cost functional (objective functional) is provided as

$$J = \int_0^T \left( A_1 I_1 + A_2 I_2 + \frac{1}{2} \sum_{i=1}^2 B_i u_i^2 \right) dt \quad (2)$$

T represents the ultimate timing needed for executing the control and  $B_i, i = 1, 2$  are positive weight constants. The cost functional (2) comprises the following; cost control measures to prevent the spread of infection among individuals, the utilization of protective gear while caring for infected patients, and lowering costs through an integrated method of pest management are all suggested by  $\frac{1}{2} B_1 u_1^2(t)$  and  $\frac{1}{2} B_2 u_2^2(t)$ . Keeping in mind that  $u_i(t) \in [0, 1], (i = 1, 2)$  and  $u_i(t) = 0$  define the absence of control efforts,  $u_i(t) = 1$  respectively, reflect the highest control effort necessary to minimize the disease. It should be emphasized that the cost function is quadratic to illustrate the non-linearity of cost implementation in nature, in accordance with [23].

As a result, it's crucial to choose an ideal control quadruple  $u^* = (u_i^*), i = 1,2$  such that

$$J(u^*) = \min\{J(u); u_i \in U\}$$

$U = \{u_1(t): 0 \leq u_i(t) \leq 1, \forall t \in [0, T]\}$  is Lebesgue measurable non-empty control set.

### 3.1. Existence of Optimal Control

The maximal principle of Pontryagin (2018), is employed to examine the existence of quadruple optimal control. The following approach is utilized by (Abidemi et al., 2022) to investigate the existence of optimal control.

**Theorem 3.1:** There exists an optimal control  $u^*$  with optimal state  $x^*$  of the system (1) satisfying

$$J(u^*) = \min\{J(u): u_i \in U\},$$

Where  $X = (S, I_1, I_2, A)$  and  $u = u_1, u_2$

The evidence for the theorem provided relies on meeting these specific criteria (Fleming et al., 1975; Okyere et al., 2020).

1. The state and control variables are non-empty
2. The control set is closed and convex
3. The right-hand side of the state system (1) is bounded by a sum of bounded control and state, and can be written as a linear function of control variable with coefficient depending on time and state.
4. The Lagrangian (integrand) of the cost functional is convex on the control set
5. There exist constants  $c_1, c_2 > 0$  and  $c_3 > 1$  such that the Lagrangian is bounded below by  $c_1(\|u\|^2)^{\frac{c_3}{2}} - c_2 > 0$

**Proof:**

**Property 1:**

The proof of property 1 is trivial since the control set  $U = \{u_1, u_2\}$  is non-empty

**Property 2:**

Given the control set  $U = [0,1]^2$ , then, by its very nature,  $U$  is considered closed. For any two arbitrary points  $y, q \in U$ , where  $y = \{y_1, y_2\}$  and  $z = \{z_1, z_2\}$ , it follows by the definition of convex set (Rectoret al., 2005) that for every  $\alpha \in [0,1]$

$\alpha y_i + (1 - \alpha)z_i \in U; i = 1,2$ . This implies that  $\alpha y_i + (1 - \alpha)z_i \in U$ . Therefore, the control set  $U$  is closed and convex.

**Property 3:** Let the right-hand side of the side of the state system (1) be denoted by  $f(t, x, u)$  where  $x = (S, I_1, I_2, A)$  and  $u = u_1, u_2$ . Then the state system (1) can be written as

$$f(t, x, u) = g(t, x) + h(t, x)u. \quad (3)$$

From (1),  $g(t, x)$  is defined by a column matrix given as

$$g(t, x) = \begin{pmatrix} \pi - \beta S(I_1 + \eta I_2) - \mu S \\ \beta S(I_1 + \eta I_2) - (\theta + \delta + \mu)I_1 \\ \theta I_1 - \mu I_2 \\ \delta I_1 - (\mu + d)A \end{pmatrix} \quad (4)$$

And  $h(t, x)$  is given by

$$\begin{pmatrix} \beta S(I_1 + \eta I_2) & 0 & 0 & 0 \\ -\beta S(I_1 + \eta I_2) & 0 & 0 & 0 \\ 0 & 0 & -\delta_1 & 0 \\ 0 & 0 & \delta_1 & 0 \end{pmatrix} \quad (5)$$

It suffices to prove that

$$\begin{aligned} \|f(t, x, u)\| &= \|g(t, x) + h(t, x)u\| \\ &\leq \|g(t, x)\| + \|h(t, x)u\| \\ &\leq \|g(t, x)\| + \|h(t, x)\| \|u\| \end{aligned} \quad (6)$$

To establish (6), the explicit algorithm presented in (Abidemiet al., 2022) is employed

**Step 1:** Find the upper bound for the column matrix  $g(t, x)$ . This is obtained as follows

$$\begin{aligned} g_{11} &= \pi \\ g_{21} &= \beta(I_1 + \eta I_2)S \\ g_{31} &= \theta I_1 \\ g_{41} &= \delta I_1 \end{aligned} \quad (7)$$

**Step2:** Squaring each entry of (7) gives

$$\begin{aligned} g_{11}^2 &= \pi^2 \\ g_{21}^2 &= \beta^2 S^2 I_1^2 + 2\beta^2 S^2 \eta I_1 I_2 + \beta^2 S^2 \eta^2 I_2^2 \\ g_{31}^2 &= \theta^2 I_1^2 \\ g_{41}^2 &= \delta^2 I_1^2 \end{aligned} \quad (8)$$

**Step3:** Sum each component in (8). Then, the resulting expression

$$\sum_1^2 g_{ij}^2 \quad (9)$$

Is obtained

**Step 4:** Find the square root of (9) which follows that (9) becomes

$$\sqrt{\sum_1^2 g_{ij}^2} \quad (10)$$

**Step 5:** Ensure each state variable in (10) is replaced by its upper bound and (Apriori boundedness) and collect like terms.  $\left(S = I_1 = I_2 = A = \frac{\pi}{\mu}\right)$

Then (10) becomes

$$\sqrt{\sum_1^2 g_{ij}^2} = \sqrt{G} \quad (11)$$

Where

$$G = \pi^2 + \beta^2 S^2 I_1^2 + 2\beta^2 S^2 \eta I_1 I_2 + \beta^2 S^2 \eta^2 I_2^2 + \theta^2 I_1^2 + \delta^2 I_1^2$$

$$G = \pi^2 + \beta^2 \frac{\pi^4}{\mu^4} + 2\beta^2 \eta \frac{\pi^4}{\mu^4} + \beta^2 \eta^2 \frac{\pi^4}{\mu^4} + \theta^2 \frac{\pi^2}{\mu^2} + \delta^2 \frac{\pi^2}{\mu^2}$$

$$G = \frac{\beta^2 \pi^4}{\mu^4} (1 + 2\eta + \eta^2) + \frac{\pi^2}{\mu^2} (\mu^2 + \theta^2 + \delta^2)$$

$$G = D_1 \pi^4 + D_2 \pi^2$$

With

$$D_1 = \frac{\beta^2}{\mu^4} (1 + 2\eta + \eta^2), D_2 = \frac{1}{\mu^2} (\mu^2 + \theta^2 + \delta^2)$$

Then

$$\sqrt{G} = \sqrt{D_1 \pi^4 + D_2 \pi^2} \quad (12)$$

**Step 6:** Using the inequality concept

$$ax + by \leq \max\{a, b\} (x + y) \quad (13)$$

This implies that

$$\sqrt{G} \leq \sqrt{\max\{D_1, D_2\} (\pi^4 + \pi^2)} \quad (14)$$

Similarly,  $h(t, x)$  is obtained by repeating steps 1-6 since  $h(t, x)$  is not a column matrix, then it follows that

**Step 2:** Squaring each of the entries of the matrix

$$h_{11}^2 = \beta(I_1 + \eta I_2)S$$

$$h_{21}^2 = -\beta(I_1 + \eta I_2)S$$

$$h_{33}^2 = -\delta_1 I_2$$

$$h_{43}^2 = \delta_1 I_2 \quad (15)$$

**Step 3:** Summing up each term in (15), then the resulting expression

$$\sum_{i=1}^2 h^2_{ij} \quad (16)$$

is obtained

**Step 4:** Finding the square root of (16), then (16) becomes

$$\sqrt{\sum_{i=1}^2 h^2_{ij}} \quad (17)$$

**Step 5:** Ensure each state variable in (17) is replaced by its upper bound (Apriori boundedness) and collect like terms

$i. e \left( S = I_1 = I_2 = A = \frac{\pi}{\mu} \right)$  Then, it follows that

$$H = \beta^2 S^2 I_1^2 + 2\beta^2 S^2 \eta I_1 I_2 + \beta^2 S^2 \eta^2 I_2^2 + \beta^2 S^2 I_1^2 + 2\beta^2 S^2 \eta I_1 I_2 + \beta^2 S^2 \eta^2 I_2^2 + \delta_1^2 I_2^2 + \delta_1^2 I_2^2$$

$$\begin{aligned}
 H &= \beta^2 \frac{\pi^4}{\mu^4} + 2\beta^2 \eta \frac{\pi^4}{\mu^4} + \beta^2 \eta^2 \frac{\pi^4}{\mu^4} + \beta^2 \frac{\pi^4}{\mu^4} + 2\beta^2 \eta \frac{\pi^4}{\mu^4} + \beta^2 \eta^2 \frac{\pi^4}{\mu^4} + \delta_1^2 \frac{\pi^2}{\mu^2} + \delta_1^2 \frac{\pi^2}{\mu^2} \\
 H &= \frac{\beta^2 \pi^4}{\mu^4} (1 + 2\eta + \eta^2 + 1 + 2\eta + \eta^2) + \frac{\pi^2}{\mu^2} (\delta_1^2 + \delta_1^2) \\
 H &= \frac{\beta^2}{\mu^4} (2 + 4\eta + 2\eta^2) \pi^4 + \frac{1}{\mu^2} (2\delta_1^2) \pi^2 \\
 H &= F_1 \pi^4 \quad , \quad H = F_2 \pi^2
 \end{aligned} \tag{18}$$

Where,

$$F_1 = \frac{\beta^2}{\mu^4} (2 + 4\eta + 2\eta^2), F_2 = \frac{1}{\mu^2} (2\delta_1^2)$$

It then follows that  $\sqrt{H} = \sqrt{F_1 \pi^4 + F_2 \pi^2}$  (19)

**Step 6:** Using the inequality concept

$$ax + by \leq \max\{a, b\} (x, y)$$

This implies that

$$\sqrt{H} \leq \sqrt{\max\{F_1, F_2\} (\pi^4 + \pi^2)}$$

Hence the proof of property 3

**Property 4:** The Lagrangian of the objective function is convex on the control set  $U$ .

Clearly, from (2), the integrand of the objective functional is of the form.

$$L(t, x, u) = A_1 I_1 + A_2 I_2 + \frac{1}{2} \sum_{i=1}^2 B_i U_i^2$$

To show that the Lagrangian is convex, that is for every arbitrary two points  $y, z \in U$

$$L(t, x, \alpha y + (1 - \alpha)z) \leq \alpha L(t, x, y) + (1 - \alpha)L(t, x, z) \tag{20}$$

For every  $\alpha \in [0,1]$

It then follows that

$$L(t, x, \alpha y + (1 - \alpha)z) = F(t, x) + \frac{1}{2} \sum_{i=1}^2 B_i (\alpha y_i + (1 - \alpha)z_i)^2 \tag{21}$$

Where  $F(t, x) = A_1 I_1 + A_2 I_2$

$$\text{Now, } \alpha L(t, x, y) = \alpha \left\{ F(t, x) + \frac{1}{2} \sum_{i=1}^2 B_i y_i^2 \right\}$$

$$= \alpha F(t, x) + \frac{\alpha}{2} \sum_{i=1}^2 B_i y_i^2 \tag{22}$$

$$(1 - \alpha)L(t, x, z) = (1 - \alpha) \left\{ F(t, x) + \frac{1}{2} \sum_{i=1}^2 B_i z_i^2 \right\}$$

$$= (1 - \alpha)F(t, x) + \frac{(1 - \alpha)^2}{2} \sum_{i=1}^2 B_i z_i^2 \tag{23}$$

Adding (22) and (23) gives

$$F(t, x) + \frac{\alpha}{2} \sum_{i=1}^2 B_i y_i^2 + \frac{(1 - \alpha)^2}{2} \sum_{i=1}^2 B_i z_i^2 \tag{24}$$

Subtract (23) from (20)

$$F(t, x) + \frac{1}{2} \sum_{i=1}^2 B_i (\alpha y_i + (1 - \alpha)z_i)^2 - F(t, x) - \frac{\alpha}{2} \sum_{i=1}^2 B_i y_i^2 - \frac{(1 - \alpha)^2}{2} \sum_{i=1}^2 B_i z_i^2$$

$$\begin{aligned}
 &= \frac{1}{2} \sum_{i=1}^2 B_i (\alpha y_i + (1 - \alpha) z_i)^2 - \frac{\alpha}{2} \sum_{i=1}^2 B_i y_i^2 - \frac{(1 - \alpha)}{2} \sum_{i=1}^2 B_i z_i^2 \\
 &= \frac{1}{2} \left\{ \sum_{i=1}^2 B_i (\alpha y_i + (1 - \alpha) z_i)^2 - \sum_{i=1}^2 B_i y_i^2 - (1 - \alpha) \sum_{i=1}^2 B_i z_i^2 \right\} \\
 &= \frac{1}{2} (\alpha^2 - \alpha) \left\{ \sum_{i=1}^2 B_i y_i^2 - 2 \sum_{i=1}^2 B_i y_i z_i + \sum_{i=1}^2 B_i z_i^2 \right\} \\
 &= \frac{1}{2} (\alpha^2 - \alpha) \sum B_i (y_i - z_i)^2 \tag{25}
 \end{aligned}$$

Since  $\alpha \in [0,1]$ , then  $\frac{1}{2} (\alpha^2 - \alpha) \sum B_i (y_i - z_i)^2 \leq 0$

It therefore follows that (25) satisfies the definition of a convex function. Rector et al., (2005). Hence, the Lagrangian is convex.

**Property 5:** There exists constant  $c_1, c_2 > 0$  and  $c_3 > 1$  such that the Lagrangian is bounded below by  $c_1(|u_i|^2)^{\frac{c_3}{2}} - c_2$

Recall,

$$L(t, x, u) = A_1 I_1 + A_2 I_2 + \frac{1}{2} \sum_{i=1}^2 B_i U_i^2 \tag{26}$$

$$\geq \frac{1}{2} \sum_{i=1}^2 B_i U_i^2$$

$$\geq \frac{1}{2} \min\{B_i\} \left( \sum_{i=1}^2 u_i^2 \right), i = 1, 2$$

$$\geq c_1(|u_i|^2)^{\frac{c_3}{2}} - c_2 \tag{27}$$

Where  $c_1 = \frac{1}{2} \min\{B_1, B_2\}$ ,  $c_2 > 0$  and  $c_3 = 2$ . This completes the proof of the existence of optimal control.

### 3.2. Characterization of Optimal Control

The Pontryagin's maximum principle is employed to derive necessary conditions for the existence of an optimal control quadruple  $(u_1^*, u_2^*)$  for the HIV/AIDS model, governed by a system of non-autonomous differential equations. By applying the Pontryagin's maximum principle, the original problem is transformed into an auxiliary minimization problem, where the objective is to minimize the Hamiltonian function with respect to the control variables.

The Hamiltonian of the optimal control problem is given by

$$H(x, u, t, \lambda) = L(x, u, t) + \lambda G(x, t, u)$$

$$\begin{aligned}
 H = & A_1 I_1 + A_2 I_2 + \frac{1}{2} \sum_{i=1}^2 B_i U_i^2 \\
 & + \lambda_S (\pi - (1 - u_1) \beta S (I_1 + \eta I_2) - \mu S) \\
 & + \lambda_{I_1} ((1 - u_1) \beta S (I_1 + \eta I_2) - (\theta + \delta + \mu) I_1) \\
 & + \lambda_{I_2} (\theta I_1 - (\mu + u_2 \delta_1) I_2) \\
 & + \lambda_A (\delta I_1 + u_2 \delta I_2 - (\mu + d) A)
 \end{aligned} \tag{28}$$

Where  $\lambda_S, \lambda_{I_1}, \lambda_{I_2}, \lambda_A$  are the adjoint variables associated with each state variable of HIV/AIDS model.

**Theorem 3.2:** Given the optimal control quadruple  $(u_1^*, u_2^*)$  that minimizes the cost functional (objective functional) over the control set  $U$  subject to the state system, then there exist adjoint variable  $\lambda_S, \lambda_{I_1}, \lambda_{I_2}, \lambda_A$ , satisfying

$$\begin{aligned}
 \frac{d\lambda_S}{dt} &= (1 - u_1) \beta (I_1 + \eta I_2) (\lambda_S - \lambda_{I_1}) + \mu \lambda_S \\
 \frac{d\lambda_{I_1}}{dt} &= (1 - u_1) \beta S (\lambda_S - \lambda_{I_1}) + \theta (\lambda_{I_1} - \lambda_{I_2}) + \delta (\lambda_{I_2} - \lambda_A) + \mu \lambda_{I_1} - A_1 \\
 \frac{d\lambda_{I_2}}{dt} &= (1 - u_1) \beta \eta S (\lambda_S - \lambda_{I_1}) + u_2 \delta_1 (\lambda_{I_2} - \lambda_A) + \mu \lambda_{I_2} - A_2 \\
 \frac{d\lambda_A}{dt} &= \lambda_A (\mu + d)
 \end{aligned} \tag{29}$$

With the transversality (final time) conditions

$$\lambda_n(T) = 0, n = S, I_1, I_2, A$$

And the optimal control characterizations

$$\begin{aligned}
 u_1^* &= \min \left\{ \max \left\{ 0, \frac{\beta S (I_1 + \eta I_2) (\lambda_{I_1} - \lambda_S)}{B_1} \right\}, 1 \right\} \\
 u_2^* &= \min \left\{ \max \left\{ 0, \frac{\delta_1 I_1 (\lambda_{I_2} - \lambda_A)}{B_2} \right\}, 1 \right\}
 \end{aligned} \tag{30}$$

**Proof:** The adjoint equations governed by the non-autonomous system (29) are obtained by taking partial derivatives of the Hamiltonian  $H$  given by (28) with respect to the associated state variables,

Differentiating the Hamiltonian  $H$  in (28) partially with respect to  $S, I_1, I_2, A$  that is,

$$\frac{d\lambda_S}{dt} = -\frac{\partial H}{\partial S}, \frac{d\lambda_{I_1}}{dt} = -\frac{\partial H}{\partial I_1}, \frac{d\lambda_{I_2}}{dt} = -\frac{\partial H}{\partial I_2}, \frac{d\lambda_A}{dt} = -\frac{\partial H}{\partial A}$$

Gives,

$$\frac{d\lambda_S}{dt} = (1 - u_1) \beta (I_1 + \eta I_2) (\lambda_S - \lambda_{I_1}) + \mu \lambda_S, \lambda_S(T) = 0 \tag{31}$$

$$\frac{d\lambda_{I_1}}{dt} = (1 - u_1) \beta S (\lambda_S - \lambda_{I_1}) + \theta (\lambda_{I_1} - \lambda_{I_2}) + \delta (\lambda_{I_2} - \lambda_A) + \mu \lambda_{I_1} - A_1, \lambda_{I_1}(T) = 0 \tag{32}$$

$$\frac{d\lambda_{I_2}}{dt} = (1 - u_1) \beta \eta S (\lambda_S - \lambda_{I_1}) + u_2 \delta_1 (\lambda_{I_2} - \lambda_A) + \mu \lambda_{I_2} - A_2, \lambda_{I_2}(T) = 0 \tag{33}$$

$$\frac{d\lambda_A}{dt} = \lambda_A(\mu + d), \lambda_A(T) = 0 \quad (34)$$

Furthermore, the control characterizations can be obtained by differentiating the Hamiltonian (H) partially with respect to  $u_1^*, u_2^*$  respectively. Then, it follows from the optimality condition

$$\frac{\partial H}{\partial u_1} = 0,$$

So that

$$\frac{\partial H}{\partial u_1} = B_1 u_1^* + \lambda_s \beta S(I_1 + \eta I_2) - \lambda_{I_1} \beta S(I_1 + \eta I_2) = 0,$$

This implies that

$$u_1^* = \frac{\beta S(I_1 + \eta I_2)(\lambda_{I_1} - \lambda_s)}{B_1} \quad (35)$$

Similarly, to obtain  $u_2^*$  using the optimality condition

$$\frac{\partial H}{\partial u_2} = 0,$$

$$\frac{\partial H}{\partial u_2} = B_2 u_2^* - \delta_1 I_2 \lambda_{I_2} + \delta_1 I_2 \lambda_A = 0$$

This implies that

$$u_2^* = \frac{\delta_1 I_2 (\lambda_{I_2} - \lambda_A)}{B_2} \quad (36)$$

Then, by standard control arguments involving bounds on the control it follows that

$$u_i^* = \begin{cases} 0, & \text{if } \pi_i^* \leq 0 \\ \pi_i & \text{if } 0 \leq \pi_i^* < 1 \\ 1 & \text{if } \pi_i^* \geq 1 \end{cases}$$

Where

$$\pi_1^* = \frac{\beta S(I_1 + \eta I_2)(\lambda_{I_1} - \lambda_s)}{B_1} \quad (37)$$

$$\pi_2^* = \frac{\delta_1 I_2 (\lambda_{I_2} - \lambda_A)}{B_2}$$

This completes the proof.

#### 4. NUMERICAL SIMULATION FOR THE OPTIMAL CONTROL PROBLEM

The optimal control problem is analyzed through numerical simulations using Matlab. The effects of incentivizing control variables, increasing awareness, and implementing various treatment strategies on the progression of HIV are evaluated. The charts will showcase two scenarios: one where only one control variable is considered while the others are set to zero, and another where two control variables are taken into account  $u_1 \neq 0, u_2 \neq 0$ . The initial human population is considered to be 500.

Optimal control strategies have a significant influence on HIV/AIDS spread in a population using time-dependent controls, public sensitization and treatment with ART was investigated for single control implementation and double control implementation to evaluate the impact of public sensitization and treatment strategies on HIV/AIDS transmission where,  $u_1 = u_2 = 0, u_1 \neq 0, u_2 = 0, u_1 = 0, u_2 \neq 0$  and  $u_1 \neq 0, u_2 \neq 0$ .

The following parameter values were obtained from relevant literatures from Table 1:

$$\pi = 300, \beta = 0.0009, \theta = 0.015, \delta_1 = 0.2, \eta = 0.3, \delta = 0.2, \mu = 0.02, d = 1.0, A_0 = 30, I_{10} = 50, I_{20} = 20, S_0 = 400, A_0 = 30, I_0 = 50, I_{20} = 20, S_0 = 400$$

In order to achieve the best results, we combine the state system, adjoint, and optimal control characterization. The HIV/AIDS model (1) was analyzed through numerical simulation using Matlab software to confirm the analytical findings and create a strong foundation for further analysis. The objectives functional weights ( $E_1 = 1, E_2 = 1, E = 1, E = 1, B_1 = 1, B_2 = 1$ ) are carefully selected to balance disease control and intervention costs. The results are presented graphically below:

#### 4.1. Optimality results

This study investigates the synergistic effects of both controls, offering critical insights into optimal HIV/AIDS management. The impact of implementing time dependent control of sensitization  $u_1(t)$  and the use of antiretroviral treatment (ART)  $u_2(t)$  on the non-autonomous system of the HIV/AIDS model are examined. The control interventions are given as follows.

##### 4.1.1. Single control implementation

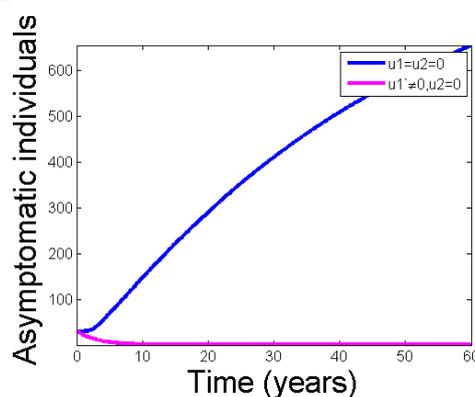
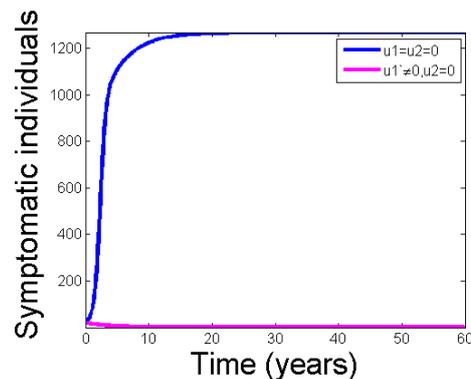


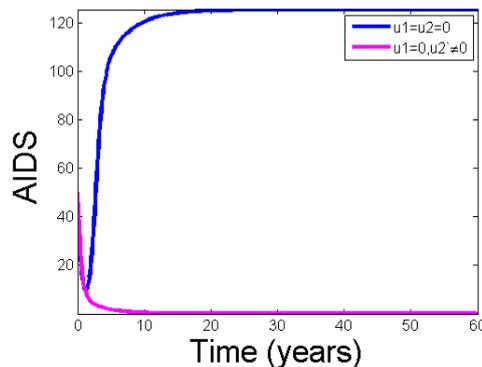
Figure 2. Behavior of Asymptomatic individuals  $I_1(t)$  with and without control

In Figure 2, the actions of individuals with no symptoms were compared with and without implementation of control measures. The population of individuals showing no symptoms decreases as the control measures  $u_1(t)$  is maintained at maximum over the period of implementation than when compared to when the control is not implemented due to careless interaction among the infective. This result is in line with Adepoju et al. (2024).



**Figure 3. Behavior of symptomatic individuals  $I_2(t)$  with and without control**

Figure 3 revealed that when antiretroviral treatment is administered to the symptomatic individuals, it reduces the viral load of infectious humans thereby suppressing the level of infectiousness. This will reduce the prevalence of the disease in the population. This is in agreement with earlier results of existing literature. See for instance (Adepoju et al., 2024; Akudibillah et al., 2018).



**Figure 4. Behavior of Full-blown Aids individuals  $A(t)$  with and without control**

Figure 4 showed the impact of use of sensitization by the full-blown AIDS individuals, there is awareness among the AIDS population about the virus and the transmission. They will certainly be more careful spreading the virus and thereby resulting to reduction in the transmission and prevalence of the disease. This aligns with the discoveries made in Adepoju et al. (2024); Chazuka et al. (2024); Sule & Abdullah (2014).

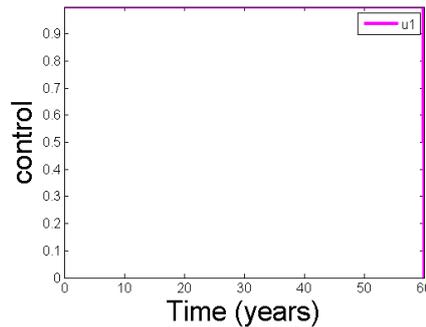


Figure 5. Control profile for the use of sensitization  $u_1(t)$

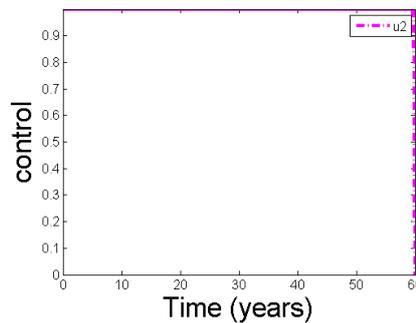


Figure 6. Control profile for the use of antiretroviral treatment  $u_2(t)$

Figure 5 showed the control profile for antiretroviral treatment  $u_1(t)$ . Antiretroviral treatment is found to be most effective within the initial 60 days of implementation, after which its effectiveness gradually decreases until reaching zero at the end of the monitoring period. Figure 6 illustrates the control strategy for promoting awareness of  $u_2(t)$ . This correlates with the results of Adepoju et al. (2024); Chazuka et al. (2024); Sule & Abdullah (2014).

#### 4.1.2. Double Control Implementation

The most effective way to prevent the transmission of HIV/AIDS among people is decided by using a combination of different strategies. The following figures show how both control combinations work.

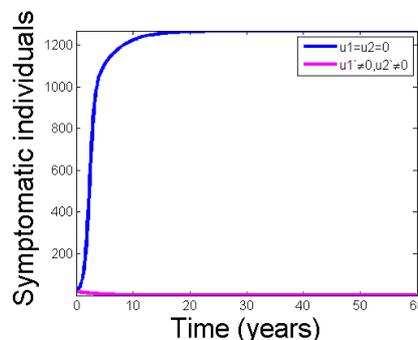
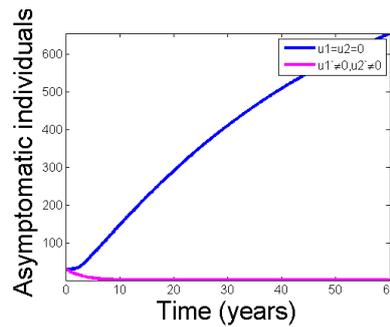
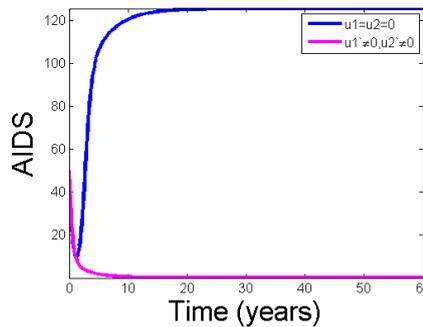


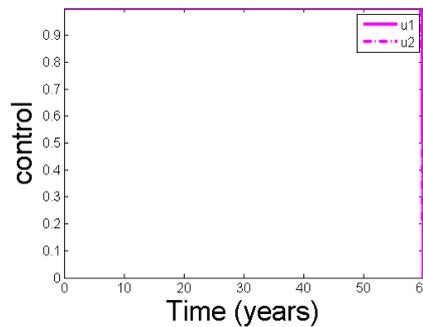
Figure 7. Behaviour of symptomatic individuals with and without optimal control  $u_1(t)$  and  $u_2(t)$



**Figure 8. Behaviour of asymptomatic individuals with and without optimal control  $u_1(t)$  and  $u_2(t)$**



**Figure 9. Behaviour of full blown AIDS individuals with and without optimal control  $u_1(t)$  and  $u_2(t)$**



**Figure 10. Control profile for the optimal control  $u_1(t)$  and  $u_2(t)$**

Figure 7 illustrates how the control combination affects those showing symptoms and taking antiretroviral treatment, as well as the use of sensitization. The data clearly shows that the  $u_1(t)$  and  $u_2(t)$  are taken into consideration, the population of symptomatic individuals decrease considerably, while in figure 8, the population of asymptomatic individuals increases without the combination of  $u_1(t)$  and  $u_2(t)$  when compared to when the optimal use of both controls are not applied. This is in agreement with earlier results of existing literature. See for instance (Adepoju et al., 2024; Akudibillah et al., 2018; Chazuka et al., 2024).

The influence of behaviour of full blown AIDS individuals with and without optimal control  $u_1(t)$  and  $u_2(t)$  as depicted in figure 9. It can be deduced that increased

investment in public sensitization campaigns can lead to significant reductions in AIDS prevalence. Policymakers should also allocate resources to sustain and scale up effective public sensitization initiatives and figure 10 showed the optimal control profile when both controls are implemented. The result is perfectly in agreement with results of (Adepoju et al., 2024).

## 5. CONCLUSION

In this work, a mathematical model which describes the transmission dynamics of HIV/AIDS is established. The existence of optimal control problem was rigorously investigated. More so, the model includes two optimal control measures which are time dependent controls: public sensitization ( $u_1 \neq 0$ ), and use of antiretroviral treatment ( $u_2 \neq 0$ ), and no treatment and public sensitization for the infective ( $u_1 = 0, u_2 = 0$ ). The effects of the control measures in combating the spread of the disease were examined. Numerical simulation results suggest that applying both effective public sensitization and treatment for the infective will definitely help in reducing the spread of HIV/AIDS in a population.

## REFERENCES

- Abidemi, A., Olaniyi, S., & Adepoju, O. A. (2022). An explicit note on the existence theorem of optimal control problem. *J. Phys. Conf. Ser.*, 2199, 12021.
- Adepoju, O. A., Ibrahim, H., & Salahu, W. O. (2024). Mathematical Assessment And Stability Analysis Of Hiv/Aids Epidemic Model With Vertical Transmission And Treatment. *Transpublika International Research In Exact Sciences*, 3(4), 1–20.
- Akudibillah, G., Pandey, A., & Medlock, J. (2018). Optimal control for HIV treatment. *Math. Biosci. Eng.*, 16(1), 373–396.
- Bräu, N., Salvatore, M., Ríos-Bedoya, C. F., Fernández-Carbia, A., Paronetto, F., Rodríguez-Orengo, J. F., & Rodríguez-Torres, M. (2006). Slower fibrosis progression in HIV/HCV-coinfected patients with successful HIV suppression using antiretroviral therapy. *Journal of Hepatology*, 44(1), 47–55.
- Chazuka, Z., Madubueze, C. E., & Mathebula, D. (2024). Modelling and analysis of an HIV model with control strategies and cost-effectiveness. *Results in Control and Optimization*, 14, 100355.
- Fauziah, I., Manaqib, M., & Zhafirah, E. M. (2024). Mathematical Modeling of HIV/AIDS Disease Spread with Public Awareness. *CAUCHY: Jurnal Matematika Murni Dan Aplikasi*, 9(1), 50–65.
- Fleming, W., Rishel, R., Fleming, W., & Rishel, R. (1975). Stochastic Differential Equations and Markov Diffusion Processes. *Deterministic and Stochastic Optimal Control*, 106–150.
- Gurmu, E. D., Bole, B. K., & Koya, P. R. (2020). Mathematical modelling of HIV/AIDS transmission dynamics with drug resistance compartment. *American Journal of Applied Mathematics*, 8(1), 34–45.

- Heath K, Levi J, Hill A. The Joint United Nations Programme on HIV/AIDS 95–95–95 targets: worldwide clinical and cost benefits of generic manufacture. *AIDS* 2021;35(1):S197–203.
- Ibrahim, I. A., Daniel, E. E., Danhaus, A. A., Adamu, M. U., Shawalu, C. J., & Yusuf, A. (2021). Mathematical Modelling of Dynamics of HIV Transmission Depicting the Importance of Counseling and Treatment. *Journal of Applied Sciences and Environmental Management*, 25(6), 893–903.
- Iddi, A. J., Massawe, E. S., & Makinde, O. D. (2012). Modelling the impact of infected immigrants on vector-borne diseases with direct transmission. *ICASTOR Journal of Mathematical Sciences*, 6(2), 143–157.
- Odebiyi, O. A., Oladejo, J. K., Elijah, E. O., Olajide, O. A., Taiwo, A. A., & Taiwo, A. J. (2024). Mathematical Modeling on Assessing the Impact of Screening on HIV/AIDS Transmission Dynamics. *Journal of Applied Sciences and Environmental Management*, 28(8), 2347–2357.
- Okosun, K. O., Makinde, O. D., & Takaidza, I. (2013). Impact of optimal control on the treatment of HIV/AIDS and screening of unaware infectives. *Applied Mathematical Modelling*, 37(6), 3802–3820.
- Okyere, E., Olaniyi, S., & Bonyah, E. (2020). Analysis of Zika virus dynamics with sexual transmission route using multiple optimal controls. *Scientific African*, 9, e00532.
- Olaniyi, S., Okosun, K. O., Adesanya, S. O., & Lebelo, R. S. (2020). Modelling malaria dynamics with partial immunity and protected travellers: optimal control and cost-effectiveness analysis. *Journal of Biological Dynamics*, 14(1), 90–115.
- Pontryagin, L. S. (2018). *Mathematical theory of optimal processes*. Routledge.
- Rabiu, M., Willie, R., & Parumasur, N. (2021). Optimal control strategies and sensitivity analysis of an hiv/aids-resistant model with behavior change. *Acta Biotheoretica*, 69, 543–589.
- Shabani, I., Massawe, E. S., & Makinde, O. D. (2011). Modelling the effect of screening on the spread of HIV infection in a population with variable inflow of infective immigrants. *Scientific Research and Essays*, 6(20), 4397–4405.
- Silva, C. J., & Torres, D. F. M. (2017). Modeling and optimal control of HIV/AIDS prevention through PrEP. *ArXiv Preprint ArXiv:1703.06446*.
- Sule, A., & Abdullah, F. A. (2014). Optimal control of HIV/AIDS dynamic: Education and treatment. *AIP Conference Proceedings*, 1605(1), 221–226.
- UNAIDS. (2002). *Paediatric HIV Infection and AIDS: UNAIDS point of view*. [https://www.unaids.org/en/resources/documents/2002/20020814\\_jc750-paediatric-pov\\_en.pdf](https://www.unaids.org/en/resources/documents/2002/20020814_jc750-paediatric-pov_en.pdf)
- van Sighem, A. I., van de Wiel, M. A., Ghani, A. C., Jambroes, M., Reiss, P., Gyssens, I. C., Brinkman, K., Lange, J. M. A., de Wolf, F., & Group, A. C. S. (2003). Mortality and progression to AIDS after starting highly active antiretroviral therapy. *Aids*, 17(15), 2227–2236.
- Zhao, Y., Elattar, E. E., Khan, M. A., Asiri, M., & Sunthrayuth, P. (2022). The dynamics of the HIV/AIDS infection in the framework of piecewise fractional differential equation. *Results in Physics*, 40, 105842.

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