COEFFICIENT INEQUALITIES OF A GENERALISED DISTRIBUTION ASSOCIATED WITH A GENERALIZED KOEBE FUNCTION

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Abstract

This study aims to explore the geometric properties of the generalized probability distribution associated with the generalized Koebe function. The inequality coefficients in the convoluted power series are evaluated using Chebyshev polynomials, with a focus on determining bounds on the initial coefficients. This research employs a mathematical-theoretic approach utilizing algebraic manipulations, theorem proving, and analytical techniques such as subordination and Hadamard products. Few conditions were set on some variables which yields certain corollaries. The results suggest that the analyzed functions have properties that are applicable in modeling complex data, allowing the description of non-normal, asymmetric, or heavy-tailed probability distributions. These findings corroborate the relevance of generalized distributions in dealing with uncertainty and variability in statistical data. Based on the existing works, the results obtained in this work are just derived. In conclusion, this study contributes to understanding the geometric relationships of generalized probability distributions. The practical implications of these findings include applications in the analysis of complex statistical data and modeling of real-world phenomena that require non-conventional distributions.

Keywords: *Univalent Functions, Convolution, Generalized Koebe Function, Chebyschev Polynomial, Subordination*

1. INTRODUCTION

Suppose B is the collection of transformation h defined as

$$h(z) = z + \sum_{m=2}^{\infty} t_m z^m \tag{1}$$

with the properties of continuity and differentiability in the unit disk $E = \{z : |z| < 1\}$ and normalized with $h(0) = h'(0) - 1 = 0$. The collection of functions with both analytic and injectivity properties are denoted by H where $H \subset B$. $S^*(\gamma)$ is the collection of H consisting of $h(z)$ which are starlike of order γ $(0 \leq \gamma < 1)$ and $h(z) \in H$ $K(\gamma)$ if $h \in H$ satisfies $zh'(z) \in S^*(\gamma)$; this mapping is said to be convex of order γ in E.

From this, it is evident that B is a superset to the collections of H while $S^*(\gamma)$ is subset of H which contain $K(\gamma)$ i.e $K(\gamma) \subset S^*(\gamma) \subset H \subset B$. Let $T \in H$ consists of the function of the form

$$f(z) = z - \sum_{m=0}^{\infty} |a_m| z^m . \tag{2}$$

which are functions with negative coefficients.

Altintas along with Adeyemo et al. (2023) looked at the class $T(\lambda, \alpha)$ $\alpha(0 \le \alpha < 1)$, $\lambda(0 \le \lambda < 1)$being the collection of members of T which satisfies the conditions stated below

$$R\left\{\frac{zh'(z)}{\lambda zh'(z) + (1-\lambda)h(z)}\right\} > \alpha;\ (z \in E) \tag{3}$$

In the same work, the class $C(\lambda, \alpha)$, $\alpha(0 \le \alpha < 1)$, $\lambda(0 \le \lambda < 1)$ being the subcollection of T comprises the assignments that validates was studied.

$$R\left\{\frac{h'(z) + zh''(z)}{h'(z) + \lambda zh''(z)}\right\} > \alpha;\qquad (z \in E) \tag{4}$$

It is deduced from (3) and (4) that $h(z) \in C(\lambda,\alpha) \Leftrightarrow zh(z) \in T(\lambda,\alpha)$. The collection $T(\lambda,\alpha)$ *and* $C(\lambda, \alpha)$ defined in (3) and (4) reduced to class of starlike function(functions which maps any point in the domain is a straight line joining any point in it to the origin lies in it) of order α, $T^*(\alpha)$ and convex function $C(\alpha)$ (sets of functions of which the lin segment joining any two points in a domain lies in it) of order α by setting $\lambda = 0$ which was studied by (Silverman, 1975). For more information on these properties, check (Adeyemo et al., 2023; Al-Ziadi and Ramadhan, 2022; Darus and Owa, 2017; Doha, 1994; Hassan and Al-Ziadi, 2023; Liu et al., 2021; Mason, 1967; Rossdy et al., 2024; Sharma et al., 2013; Wanas and Ahsoni, 2022).

Recently, (Altintas and Owa, 1988) gave a more generalized form of the function

$$h(z) = \frac{z}{1-z^2} = z + z^3 + z^5 + \ldots \qquad z \in E \tag{5}$$

which is the class $S^*(0) \equiv S^*$ and $K(0) \equiv K$ as

$$h_\alpha(z) = \frac{z}{1-z^\alpha} = z + \sum_{m=2}^{\infty} t_m z^{1+m\alpha} \tag{6}$$

for some real $\alpha(0 < \alpha \le 2)$. Some properties between functions $h(z)$ in (3) and (4) were discussed using the principal value for $z^{k\alpha}$.

The collection A_α with analytic transformation $h(z)$ with definite series that has the template

$$h(z) = z + \sum_{m=2}^{\infty} t_m z^{1+m\alpha} \qquad z \in E \tag{7}$$

for some real $\alpha(0 < \alpha \le 2)$ was introduced.

A more generalized form of the function defined in (6) and (7) is being defined as follows

$$h_{\alpha,\mu} = \frac{z}{(1-z^\alpha)^\mu} = z + \sum_{m=2}^{\infty} \frac{(\mu+m-2)!}{(\mu-1)!(m-1)!} z^{1+(m-1)\alpha} \qquad z \in E \tag{8}$$

for some real α and μ, $\alpha(0 < \alpha \le 1)$ and $\mu \ge 1$ serve as the basis for this present work. Equation (8) reduces to (5) when $(\alpha = 2,\ \mu = 1)$and reduces to (6) when $(\mu = 1)$.

Probability distributions are fundamentals tools in statistical modelling, enabling the characterization of uncertainty and variability in complex systems. Pobability distributions, such as the normal and exponential distributions, often fails to adequately capture the intricacies of real world data. To bridge this gap, generalized probability distributions serves as a versatile and powerful tools for modelling complex phenomenal.

The means of generalized distribution which was recently examined by Abiodun Tinuoye Oladipo (2016), (2019); and Porwal (2014) was used to establish some properties related to (5), (6), (7) and (8).

Let $S = \sum_{n+0}^{\infty} a_n$ such that a_n is non negative($a_n \geq 0$) $\forall$ $n \in N$ is convergent. The generalized discrete probability distribution whose mass function is given as

$$P(n) = \frac{a_n}{S}, \; n = 0, 1, 2, 3, 4 \; \ldots \quad (9)$$

(9) is a probability mass function because each probability is non negative($p_n \geq 0$) and the sum of the probabilities equals 1 $\sum P(n) = 1$. From $S = \sum_{n=0}^{\infty} a_n$, the authors in Oladipo & Opoola (2010) and Porwal (2013) (2018) defined the series

$$\phi(x) = \sum_{n=0}^{\infty} a_n x^n \quad (10)$$

which is convergent for $|x| < 1$ and for $x = 1$, see for detail in (Oladipo & Opoola, 2010; 2016; and Porwal, 2014) which are cited in the reference. The expected values $E(X)$ of a distinct random variable $X(x_1, x_2, x_3, \ldots)$ with probabilities $p_1, p_2, p_3\, p_4 \ldots$ is defined by

$$E[X] = \sum_{n=0}^{\infty} p_n X_n$$

and the rth moment about $X = 0$ is defined by n

$$\mu'_r = E(X^r)$$

Where μ'_1 is the mean of the distribution and variance of the distribution is given by

$$\mu'_2 - (\mu'_1)^2.$$

The moment about the origin is defined as

$$a.\; \mu'_1 = \sum_{n=0}^{\infty} np(n) = \sum_{n=0}^{\infty} n\frac{a_n}{S} = \frac{1}{S}\sum_{n=0}^{\infty} na_n = \frac{\phi'(1)}{S}$$

$$b. \mu'_2 = \sum_{n=0}^{\infty} n^2 \frac{a_n}{S} = \frac{1}{S}[\sum_{n=0}^{\infty} n(n-1)a_n + \sum_{n=1}^{\infty} na_n] = \frac{1}{S}[\phi''(1) + \phi'(1)]$$

Thus, the mean and variance of the distribution are respectively defined as

$$mean = \mu'_1 = \frac{\phi'(1)}{S}$$

$$Variance = \mu'_2 - (\mu'_1)^2 = \frac{1}{S}[\phi''(1) + \phi'(1) - \frac{(\phi'(1))^2}{S},$$

See Porwal (2018) and Oladipo (2020) for further discussion.

A power series of the form

$$K_{\phi}(z) = z + \sum_{m=2}^{\infty} \frac{a_{m-1}}{S} z^{m} \tag{11}$$

is of interest in this present work. Distinct Hadamand product or rotation of $h(z)$ defined by (1) and $d(z)$ with the Taylor's series expansion $d(z) = z + b_2z^2 + b_3z^3 + \ldots$ denoted $h{*}d$ is given by

$$(h \star d)(z) = z + \sum_{m=2}^{\infty} a_m b_m z^{m} \tag{12}$$

A subcollection $A_{\alpha,\mu}$ of analytic transformation with series form of

$$T_{\alpha,\mu,z} = z + \sum_{m=2}^{\infty} \frac{(\mu + m - 2)!}{(\mu-1)!(m-1)!} \frac{a_{m-1}}{S} z^{1+(m1)\alpha} \tag{13}$$

is being investigated using (Mostafa, 2009; Oladipo & Opoola, 2010; and Oladipo, 2016).

Based on the above background, this study is interesting to investigate because it offers further development of generalized probability distributions, which have an important role in modeling complex data. Conventional probability distributions are often unable to capture the complexity of real data, such as asymmetric or heavy-tailed distributions. To address this challenge, this study examines the geometric properties of generalized distributions with a novel approach using Chebyshev polynomials. The main focus of the research is to review the initial coefficients on the convoluted power series and provide bounds on their inequalities, which is an area that has not been widely explored in previous literature. By relating the generalized probability distribution to the Koebe function, this research aims to make a significant contribution in modeling complex data that includes non-normal, asymmetric, or heavy-tailed distributions.

2. RESEARCH METHODS

2.1. Research Type

This research is theoretical and oriented towards pure mathematics. The focus is on the development and analysis of mathematical properties of certain functions, such as generalized probability distributions, Koebe functions, and Chebyshev polynomials.

2.2. Research Approach

The approach used is mathematical analysis. This research involves algebraic manipulation, theorem proving, and exploration of the properties of functions through mathematical transformations.

2.3. Data Analysis Technique

a) Coefficient Analysis: Determines the inequality of coefficients on defined functions.

b) Use of Chebyshev Polynomials: Chebyshev polynomials are used to evaluate the initial boundaries of the coefficients.

c) Subordination and Hadamard Products: This technique is applied to explore the relationship between functions in a given domain.
d) Use of Univalent Functions: Univalent functions are used to identify the geometric relationship between the analytic functions studied.

Lemma [1]: A function denoted as

$$h(z) = z - \sum_{m=2}^{\infty} |a_m| z^m$$

is in the class $T(\lambda,\alpha)$ if and only if

$$\sum_{m=2}^{\infty} [m - \alpha\lambda m - \alpha + \lambda\alpha] |a_m| \leq 1 - \alpha$$

Lemma [1]: A function

$$h(z) = z - \sum_{m=2}^{\infty} |a_m| z^m$$

is in the collection $C(\lambda,\alpha)$ if and only if

$$\sum_{m=2}^{\infty} m[m - \lambda\alpha m - \alpha + \lambda\alpha] |a_m| \leq 1 - \alpha.$$

3. RESULTS AND DISCUSSION

3.1. Main Results

Theorem 3.1 :Let h be of form (13), then $h \in T(\lambda,\alpha)$ if and only if

$$\left| \sum_{m=2}^{\infty} \frac{(\mu+m-2)!}{(\mu-1)!(m-1)!} \left[(1-\beta\lambda)\psi'(1) + (1-\beta)(\psi(1) - \psi(0)) \right] \left| \frac{a_{m-1}}{S} \right| \right| \leq 1-\beta$$

Proof : From (3) and (13) we have

$$h(z) = z + \sum_{m=2}^{\infty} \frac{(\mu+m-2)!}{(\mu-1)!(m-1)!} \frac{a_{m-1}}{S} z^{1+(m-1)\alpha}$$

$$zh'(z) = z + \sum_{m=2}^{\infty} \frac{(\mu+m-2)!}{(\mu-1)!(m-1)!} (1+(m-1)\alpha) \frac{a_{m-1}}{S} z^{1+(m-1)\alpha}$$

$$\lambda zh'(z) = \lambda z + \sum_{m=2}^{\infty} \frac{(\mu+m-2)!}{(\mu-1)!(m-1)!} (1+(m-1)\alpha) \frac{a_{m-1}}{S} z^{1+(m-1)\alpha}$$

$$(1-\lambda)h(z) = (1-\lambda)z + (1-\lambda) \sum_{m=2}^{\infty} \frac{(\mu+m-2)!}{(\mu-1)!(m-1)!} \frac{a_{m-1}}{S} z^{1+(m-1)\alpha}$$

$$\frac{zh'(z)}{zh'(z)+(1-\lambda)h(z)} \le \beta \Rightarrow$$

$$1-\beta \ge \sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!!}[\beta\lambda(1+(m-1)\alpha)+\beta-\beta\lambda-(1+(m-1))\alpha]\frac{a_{m-1}}{S}$$

$$\ge \sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}[(1+(m-1)\alpha)(\beta\lambda-1)+\beta(1-\lambda)]\frac{a_{m-1}}{S}$$

$$\ge \sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}[[1+(m-1)\alpha-1](\beta\lambda-1)+(\beta-1)]\frac{a_{m-1}}{S}$$

$$\ge \sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}[(m-1)\alpha(\beta\lambda-1)+(\beta-1)]\frac{a_{m-1}}{S}$$

$$= \sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}\frac{1}{S}[(\beta\lambda-1)(m-1)\alpha\, a_{m-1}+(\beta-1)]a_{m-1}$$

$$= \sum_{m=1}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}\frac{1}{S}[(\beta\lambda-1)m\alpha\, a_m+(\beta-1)a_m]$$

$$= \sum_{m=1}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}\frac{1}{S}[(\beta\lambda-1)\phi'(1)+(\beta-1)[\phi(1)-\phi(0)]]a_m|$$

Thus, $$\left|\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}[(1-\beta\lambda)\psi'(1)+(1-\beta)(\psi(1)-\psi(0))]\left|\frac{a_{m-1}}{S}\right|\right| \le 1-\beta$$

The table below gives the consequences of the result by setting some conditions

Table 1. Consequences of the result in certain conditions

m	$\alpha=\mu$	$\lambda=\beta$	$\psi(1)$	$\psi'(1)$	$\frac{a_{m-1}}{S}=\frac{a_1}{S}$	$\frac{a_{m-1}}{S}=\frac{a_2}{S}$
2	1	0	1	2	$\frac{1}{3}$	
		0.25			$\frac{2}{7}$	
		0.5			$\frac{1}{4}$	
		0.75			$\frac{2}{9}$	
	2	0	2	6	$\frac{1}{16}$	
		0.25			$\frac{1}{19}$	

		0.5			$\frac{1}{22}$	
		0.75			$\frac{1}{25}$	
3	1	0	1	3		$\frac{1}{4}$
		0.25				$\frac{4}{19}$
		0.5				$\frac{2}{11}$
		0.75				$\frac{4}{25}$
	2	0	0			$\frac{1}{54}$
		0.25	0.25			$\frac{4}{261}$
		0.5	0.5			$\frac{2}{153}$
		0.75	0.75			$\frac{4}{351}$

Corollary 3.2 : Let λ = 0 in theorem 3.1 then

$$\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}\left[\psi'(1)+(1-\beta)(\psi(1)-\psi(0))\right]\frac{a_{m-1}}{S}\leq 1-\beta \tag{14}$$

Corollary 3.3 : Let λ = 1 in theorem (3.1), we have

$$\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}\left[(1-\beta)\psi'(1)+(1-\beta(\psi(1)-\psi(0)))\right]\frac{a_{m-1}}{S}\leq 1 \tag{15}$$

Theorem 3.2 : If $T(\mu,\alpha,z)$ is of the form (13) is in class $C(\lambda,\alpha)$ if and only if

$$\left|\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}\left[(1-\beta\lambda)\psi''(1)+(3-2\beta\lambda-\beta)\psi'(1)+(1-\beta)(\psi(1)-\psi(0))\right]\right|\left|\frac{a_{m-1}}{S}\right|\leq 1-\beta \tag{16}$$

Proof : From(4) and (13) we have

$$h(z)\ \ z+\sum_{m=2}^{\infty}\frac{(\mu+m+-2)!)}{(\mu-1)!(mk-1)!}\frac{a_{m-1}}{S}z^{1+(m-1)\alpha}$$

$$h'(z)=1+\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}\frac{a_{m-1}}{S}z^{(m-1)\alpha}$$

$$h''(z)=\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}(m-1)\alpha(1+(m-1)\alpha)\frac{a_{m-1}}{S}z^{(m-1)\alpha-1}$$

$$\lambda zh''(z)=\lambda\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}(m-1)\alpha(1+(m-1)\alpha)\frac{a_{mk-1}}{S}z^{(m-1)\alpha}$$

$$\frac{h'(z)\ +zh''(z)}{h'(z)+\lambda zh''(z)}>\beta$$

$$\frac{1+\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}[1+(m-1)\alpha]\frac{a_{m-1}}{S}z^{(m-1)\alpha}+\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}[1+(m-1)\alpha][(m-1)\alpha]\frac{a_{m-1}}{S}z^{(m-1)\varepsilon}}{1+\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!)(m-1)!}[1+(m-1)\alpha]\frac{a_{m-1}}{S}z^{(m-1)\alpha}+\lambda\sum_{m=}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}[1+(m-1)\alpha[(m-1)\alpha]]\frac{a_{m-1}}{S}z^{(m-1)\alpha}}\leq\beta$$

$$\frac{1+\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}[1+(m-1)\alpha+(1+(m-1)\alpha)(m-1)\alpha]\frac{a_{m-1}}{S}z^{(m-1)\alpha}}{1+\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}[1+(m-1)\alpha\ +\ \lambda(1+(m-1)\alpha)(m-1)\alpha]\frac{a_{m-1}}{S}z^{(m-1)\alpha}}$$

$As\ \ z\rightarrow 1^{-}$

$$1-\beta\geq\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}\begin{bmatrix}\beta[(1+(m-1)\alpha)+\lambda(1+(m-1)\alpha)(m-1)\alpha]- \\ [(1+(m-1)\alpha)+(1+(m-1)\alpha)(m-1)\alpha]\end{bmatrix}\frac{a_{m-1}}{S}$$

$$\geq-\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}[1+(m-1)\alpha)(m-1)\alpha+(1+(m-1)\alpha)-\beta(1+(m-1)\alpha)-\beta\lambda(1+(m-1)\alpha)(m-1)\alpha]\frac{a_{m-1}}{S}$$

$$\geq-\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}[(1+(m-1)\alpha)^{2}\ -\beta(1+(m-1)\alpha)-\beta\lambda(1+(m-1)\alpha)^{2}+\beta\lambda(1+(m-1)\alpha)]\frac{a_{m-1}}{S}$$

$$\geq-\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}(1+(m-1)\alpha)[(1+(m-1)\alpha)+\beta\lambda-\beta\lambda(1+(m-1)\alpha)-\beta]\frac{a_{m-1}}{S}$$

$$\geq-\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}(1+(m-1)\alpha)[(1+(m-1)\alpha)(1-\beta\lambda)+\beta\lambda-\beta]\frac{a_{m-1}}{S}$$

$$\geq-\sum_{m=1}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}(1+(m-1)\alpha)[(1+(m-1)\alpha)(1-\beta\lambda)-\beta(1-\lambda)]\frac{a_{m-1}}{S}$$

$$\geq-\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}\begin{bmatrix}(1-\beta\lambda)(1+(m-1)\alpha-1)(1+(m-1)\alpha-2) \\ +(3-2\beta\lambda-\beta)(1+(m-1)\alpha-1)+(1-\beta)\end{bmatrix}\frac{a_{m-1}}{S}$$

$$\geq-\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}[(1-\beta\lambda)\psi''(1)\ +(3-2\beta\lambda-\beta)\psi'(1)\ +(1-\beta)(\psi(1)-\psi(0))]$$

Thus,

$$\left|\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}[(1-\beta\lambda)\psi''(1)\ +(3-2\beta\lambda-\beta)\psi'(1)+(1-\beta)(\psi(1)-\psi(0))]\right|\left|\frac{a_{m-1}}{S}\right|\leq 1-\beta$$

Table 2. Consequences of the result in certain conditions

m	$\alpha=\mu$	$\lambda=\beta$	$\psi(1)$	$\psi'(1)$	$\psi''(1)$	$\frac{a_{m-1}}{S}=\frac{a_1}{S}$	$\frac{a_{m-1}}{S}=\frac{a_2}{S}$
2	1	0	1	2	2	$\frac{1}{9}$	
		0.25				$\frac{2}{21}$	
		0.5				$\frac{1}{12}$	
		0.75				$\frac{2}{27}$	
	2	0	2	6	12	$\frac{1}{64}$	
		0.25				$\frac{1}{76}$	
		0.5				$\frac{1}{88}$	
		0.75				$\frac{1}{100}$	
3		0	1	3	6		$\frac{1}{16}$
		0.25					$\frac{1}{19}$
		0.5					$\frac{1}{22}$
		0.75					$\frac{1}{25}$
	2	0	3	15	60		$\frac{1}{324}$
		0.25					$\frac{2}{783}$
		0.5					$\frac{1}{459}$
		0.75					$\frac{2}{1053}$

Corollary 3.3: Let $\lambda = 0$ then

$$\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}[\psi''(1)+(3-\beta)\psi'(1)+(1-\beta)(\psi(1)-\psi(0))]\frac{a_{m-1}}{S}\leq 1-\beta$$

Corollary 3.4: Let $\lambda = 1$ then
$$\sum_{m=2}^{\infty}\frac{(\mu+m-2)!}{(\mu-1)!(m-1)!}\left[\psi''(1)+3\psi'(1)+\psi(1)-\psi(0)\right]\frac{a_{m-1}}{S}\leq 1$$

4. COEFFICIENT ESTIMATES

Two analytic functions in a domain U are subordinated to each other, say, h, to d represented by $h \prec d$ if there exist a relation $w(z)$ that is a smooth member of the domain and has the properties $w(0) = 0$ and $|w(z)| < 1$ such that $h(z) = d(w(z))$ $(z \in U)$. Such a relation $w(z)$ is called Scharwz function.

One of the important tools in numerical analysis is Chebyshev Polynomial which is of different forms and used in different ways. The form of Chebyshev polynomial of interest in this work is the form $G_j(t) = \cos jt$ and $v_j(t) = \frac{\sin(j+1)t}{\sin t}$ where $t = \cos\theta$ and the polynomial degree is denoted by j. For further studies on Chebyshev polynomial see Doha (1994) and Mason (1967).

A relation h in B is in the collection $C(\mu, \kappa, \alpha; t)$, $\mu > 1$, $\kappa, \alpha > 0$ and $t \in (-\frac{1}{2}, 1]$ if the following subordinations hold

$$(1-\kappa)\frac{zh'(z)}{h(z)}+\kappa(1+\frac{zh''(z)}{h'(z)}) \prec G(z,t) := \frac{1}{1-2tz+z^2} \qquad (z \in U)\ (17)$$

and $G(z,t) = 1 + 2\cos\beta z + (3\cos^2\beta - \sin^2\beta)z^2 + \ldots$ $z \in U$ if $t = \cos\beta$ and $\beta \in (-\frac{\pi}{3}, \frac{\pi}{3})$.

Following Whittaker and Watson (1963), $G(z,t)$ can be re notated as
$G(z,t) = 1 + V_1(t)z + V_2(t)z^2 + V_3(t)z^3 + \ldots (z \in U, t \in (-1,1))$

where $V_{j-1}(t) = \frac{\sin(j\arccos t)}{\sqrt{1-t^2}}$ $j \in N$ are one of the forms of Chebyshev Polynomial. Recursively, $V_j(t) = 2tV_{j-1}(t) - V_{k-2}(t)$. This gives
$V_1(t) = 2t$
$V_2(t) = 4t^2 - 1$
$V_3(t) = 8t^3 - 4t$

Theorem 4.1 : Let the function $T(\mu,\alpha,z)$ given by (13)be in the class $C(\mu,\alpha,\kappa,t)$ then

$$\left|\frac{a_1}{S}\right| \leq \frac{2t}{\mu\alpha(1+\alpha\kappa)} \quad (18)$$

$$\left|\frac{a_2}{S}\right| \leq \frac{[2\alpha t(1+\alpha\kappa)^2+4\alpha t^2(1+\alpha\kappa)^2-\alpha(1+\alpha\kappa)^2+4\kappa(1+2\alpha t)}{2\alpha^2\mu(\mu+1)(1+2\alpha)(1+\alpha\kappa)^2}$$

$$\left|\frac{a_3}{S}\right| \leq \frac{2[(8t^3+4t^2-2t)(\alpha^3(1+\alpha\kappa)^3)-8t^3(1+\alpha)-8\kappa(1+\alpha)-Q(3\alpha+2\kappa-\kappa(1+\alpha)}{3\alpha^3\mu(\mu+1)(\mu+2)(1+\alpha\kappa)^3} \quad (19)(20)$$

where

$$Q = \frac{t\left[2\alpha t(1+\alpha\kappa)^2+4\alpha t^2(1+\alpha\kappa)^2-\alpha(1+\alpha\kappa)^2+4\kappa t(1+2\alpha+\alpha^2)\right]}{\alpha^3(1+2\alpha)(1+\alpha\kappa)^3}$$

Proof : From (17) and (13) we have

$$1 + \left(\mu(1+\alpha)\frac{a_1}{S} - \mu\frac{a_1}{S}\right)z^\alpha + \left(\mu^2\frac{a_1^2}{S^2} - \mu(\mu+1)\frac{a_1}{S} - \mu^2\frac{a_1^2}{S^2}(1+\alpha) + \mu(\mu+1)\frac{a_2}{S}(1+2\alpha)\right)z^{2\alpha}$$
$$+ \left(2\mu^2(\mu+1)\frac{a_1a_2}{S^2} - \mu(\mu+1)\frac{(\mu+2)}{2}\frac{a_3}{S} - \mu^2(\mu+1)(1+\alpha)\frac{a_1a_2}{S^2} + \mu^3\frac{a_1^3}{S^3}(1+\alpha)\right.$$
$$\left. - \mu^2(\mu+1)(1+2\alpha)\frac{a_1a_2}{S^2} + \mu(\mu+1)\frac{(\mu+2)}{2}(1+3\alpha)\frac{a_3}{S}\right)z^{3\alpha} + \dots$$

$$\frac{1 + \mu\frac{a_1}{S}(1+\alpha)^2 z^\alpha + \mu(\mu+1)\frac{a_2}{S}(1+2\alpha)^2 z^{2\alpha} + \mu(\mu+1)\frac{(\mu+2)}{2}\frac{a_3}{S}(1+3\alpha)^2 z^{3\alpha}}{1 + \mu\frac{a_1}{S}(1+\alpha) z^\alpha + \mu(\mu+1)\frac{a_2}{S}(1+2\alpha) z^{2\alpha} + \mu(\mu+1)\frac{(\mu+2)}{2}\frac{a_3}{S}(1+3\alpha) z^{3\alpha}}$$

$$\mu\frac{a_1}{S}\left(1 + \alpha - 1 - \kappa(1+\alpha) + \kappa + \kappa(1+\alpha)^2 - \kappa(1+\alpha)\right)z^\alpha$$
$$+ \left(\mu^2\frac{a_1^2}{S^2}(1+\alpha) + \mu(\mu+1)\frac{a_2}{S}(1+2\alpha) - \kappa\mu^2\frac{a_1^2}{S^2} + \kappa\mu(\mu+1)\frac{a_2}{S} + \kappa\mu^2(1+\alpha)\frac{a_1^2}{S^2}\right.$$
$$- \kappa\mu(\mu+1)(1+2\alpha)\frac{a_2}{S} + \kappa\mu^2(1+\alpha)^2\frac{a_1^2}{S^2} - \kappa\mu(\mu+1)(1+2\alpha)\frac{a_2}{S} - \kappa\mu^2(1+\alpha)^3\frac{a_1^2}{S^2}$$
$$\left. + \kappa\mu(\mu+1)(1+2\alpha)^2\frac{a_2}{S}\right)z^{2\alpha} + \left(2\mu^2(\mu+1)\frac{a_1a_2}{S^2} - \mu^2(\mu+1)(1+\alpha)\frac{a_1a_2}{S^2} + \mu^3(1+\alpha)\frac{a_1^3}{S^3}\right.$$
$$- \mu^2(\mu+1)(1+2\alpha)\frac{a_1a_2}{S^2} - 2\kappa\mu^2(\mu+1)\frac{a_1a_2}{S^2} + \kappa\mu^2(\mu+1)(1+\alpha)\frac{a_1a_2}{S^2} + \kappa\mu^3(1+\alpha)\frac{a_1^3}{S^3}$$
$$+ \kappa\mu^2(\mu+1)(1+2\alpha)\frac{a_1a_2}{S^2} + \kappa\mu^2(\mu+1)(1+2\alpha)\frac{a_1a_2}{S^2} + 2\kappa\mu^2(\mu+1)(1+\alpha)(1+2\alpha)\frac{a_1a_2}{S^2}$$
$$+ \kappa\mu^2(\mu+1)(1+\alpha)(1+2\alpha)\frac{a_1a_2}{S^2} - \kappa\mu^2(\mu+1)(1+\alpha)(1+2\alpha)\frac{a_1a_2}{S^2}$$
$$+ \kappa\mu(\mu+1)\frac{(\mu+2)}{2}(1+3\alpha)^2\frac{a_3}{S} - \mu(\mu+1)\frac{(\mu+2)}{2}\frac{a_3}{S} + \mu(\mu+1)\frac{(\mu+2)}{2}(1+3\alpha)\frac{a_3}{S}$$
$$\left. + \kappa\mu(\mu+1)\frac{(\mu+2)}{2}\frac{a_3}{S} - \kappa\mu(\mu+1)\frac{(\mu+2)}{2}(1+3\alpha)\frac{a_3}{S}\right)z^{3\alpha}$$

Comparing with the Chebyschev polynomial of first kind, we have

$$\mu\frac{a_1}{S}\left(1 + \alpha - 1 - \kappa(1+\alpha) + \kappa + \kappa(1+\alpha)^2 - \kappa(1+\alpha)\right)z^\alpha = V_1(t)c_1 \tag{21}$$

$$\frac{a_1}{S} = \frac{2t}{\mu\alpha(1+\alpha\kappa)} \tag{22}$$

$$\left(\mu^2\frac{a_1^2}{S^2}(1+\alpha) + \mu(\mu+1)\frac{a_2}{S}(1+2\alpha) - \kappa\mu^2\frac{a_1^2}{S^2} + \kappa\mu(\mu+1)\frac{a_2}{S} + \kappa\mu^2(1+\alpha)\frac{a_1^2}{S^2}\right.$$
$$- \lambda\mu(\mu+1)(1+2\alpha)\frac{a_2}{S} + \kappa\mu^2(1+\alpha)^2\frac{a_1^2}{S^2} - \kappa\mu(\mu+1)(1+2\alpha)\frac{a_2}{S} - \kappa\mu^2(1+\alpha)^3\frac{a_1^2}{S^2}$$
$$\left. + \kappa\mu(\mu+1)(1+2\alpha)^2\frac{a_2}{S}\right)z^{2\alpha} = V_1(t)c_2 + V_2(t)c_1^2$$

On simplification and using (22) gives

$$\frac{a_2}{S} = \frac{[2\alpha t(1+\alpha\kappa)^2 + 4\alpha t^2(1+\alpha\kappa)^2 - \alpha(1+\alpha\kappa)^2 + 4\kappa(1+2\alpha t)}{2\alpha^2\mu(\mu+1)(1+2\alpha)(1+\alpha\kappa)^2} \tag{23}$$

$$\Big(2\mu^2(\mu + 1)\frac{a_1a_2}{S^2} - \mu^2(\mu+1)(1+\alpha)\frac{a_1a_2}{S^2} + \mu^3(1+\alpha)\frac{a_1^3}{S^3}$$
$$- \mu^2(\mu+1)(1+2\alpha)\frac{a_1a_2}{S^2} - 2\kappa\mu^2(\mu+1)\frac{a_1a_2}{S^2} + \kappa\mu^2(\mu+1)(1+\alpha)\frac{a_1a_2}{S^2} + \kappa\mu^3(1+\alpha)\frac{a_1^3}{S^3}$$
$$+ \kappa\mu^2(\mu+1)(1+2\alpha)\frac{a_1a_2}{S^2} + \kappa\mu^2(\mu+1)(1+2\alpha)\frac{a_1a_2}{S^2} + 2\kappa\mu^2(\mu+1)(1+\alpha)(1+2\alpha)\frac{a_1a_2}{S^2}$$
$$+ \kappa\mu^2(\mu+1)(1+\alpha)(1+2\alpha)\frac{a_1a_2}{S^2} - \kappa\mu^2(\mu+1)(1+\alpha)(1+2\alpha)\frac{a_1a_2}{S^2}$$
$$+ \kappa\mu(\mu+1)\frac{(\mu+2)}{2}(1+3\alpha)^2\frac{a_3}{S} - \mu(\mu+1)\frac{(\mu+2)}{2}\frac{a_3}{S} + \mu(\mu+1)\frac{(\mu+2)}{2}(1+3\alpha)\frac{a_3}{S}$$
$$+ \kappa\mu(\mu+1)\frac{(\mu+2)}{2}\frac{a_3}{S} - \kappa\mu(\mu+1)\frac{(\mu+2)}{2}(1+3\alpha)\frac{a_3}{S}\Big)z^{3\alpha} = V_1(t) + V_2(t) + V_3(t)$$

Using (22) and (23) in the above gives

$$\frac{a_3}{S} = \frac{2[(8t^3+4t^2-2t)(\alpha^3(1+\alpha\kappa)^3) - 8t^3(1+\alpha) - 8\kappa(1+\alpha) - Q(3\alpha+2\kappa-\kappa(1+\alpha)}{3\alpha^3\mu(\mu+1)(\mu+2)(1+\alpha\kappa)^3} \tag{24}$$

where

$$Q = \frac{t\Big[2\alpha t(1+\alpha\kappa)^2 + 4\alpha t^2(1+\alpha\kappa)^2 - \alpha(1+\alpha\kappa)^2 + 4\kappa t(1+2\alpha+\alpha^2)\Big]}{\alpha^3(1+2\alpha)(1+\alpha\kappa)^3} \tag{25}$$

Corollary 4.1**:** Let $\kappa = 0$ in theorem (4.1) then

$$|\frac{a_1}{S}| \leq \frac{2t}{\mu\alpha}$$

$$|\frac{a_2}{S}| \leq \frac{4t^2+2t-1}{2\alpha\mu(\mu + 1)(1+2\alpha)}$$

$$|\frac{a_3}{S}| \leq \frac{\alpha^4(1+2\alpha)(8t^3+4t^2-2t) - \alpha(1+\alpha)(1+2\alpha)8t^3 - (12t^3+6t^2-3)}{3\alpha^4(1+2\alpha)\mu(\mu+1)(\mu+2)}$$

Corollary 4.2: Let $\alpha = \frac{1}{2}$ then

$$|\frac{a_1}{S}| \leq \frac{8t}{(2+\kappa)\mu}$$

$$|\frac{a_2}{S}| \leq \frac{4+4\kappa+\kappa^2}{4}\frac{(4t^2+2t-1)}{4\mu(\mu+1)}$$

$$|\frac{a_3}{S}| \leq \frac{8\{16(8t^3+4t^2-2t)(\frac{\kappa+2}{4})^3 - 192t^3 - 192\kappa - [(3+\kappa)t\{(\kappa+2^2(2t+4t^2-1)+72\kappa t\}\}}{3\mu(\mu+1)(\mu+2)(\kappa+2)^3}$$

Corollary 4.3: Let $\kappa = 0 \; and \; \alpha = \frac{1}{2}$ then

$$|\frac{a_1}{S}| \leq \frac{4t}{\mu}$$

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$$|\frac{a_2}{S}| \leq \frac{4t^2+2t-1}{4\mu(\mu+1)}$$

$$|\frac{a_3}{S}| \leq \frac{176t^3-88t^2+28t}{3\mu(\mu+1)(\mu+2)}$$

This research led to key findings focusing on the analysis of the initial coefficients of the power series formed through the convolution of the generalized probability distribution with the generalized Koebe function. Using Chebyshev polynomials, this research successfully determines the boundaries of the coefficient inequality and provides a new formulation for a more flexible probability distribution. The results show that this formulation is able to capture complex characteristics, such as asymmetric or heavy-tailed distributions, which often cannot be modeled by conventional probability distributions. In addition, this study also establishes important corollaries by providing special forms of the results for certain conditions, such as the parameters λ and α, which simplifies the modeling in certain cases.

The analysis shows that this approach makes a significant contribution to understanding the geometric relationships between analytic functions, particularly in the domain of univalent functions. The implications of these results are highly relevant for modeling complex data, including data with non-normal properties often found in various disciplines, such as statistics, physics, and social sciences. By extending the generalization of the Koebe function and redefining the coefficient bounds, this research offers a more comprehensive mathematical framework to handle analytical challenges in probability distribution modeling. This opens up opportunities for further development in data analysis covering non-conventional distributions and applications to real-life scenarios.

The implication of this research is that it makes an important contribution to the field of mathematical modeling, particularly for data that is difficult to model using standard probability distributions. The results can be applied in a variety of disciplines, including statistics, physics, and social sciences, where complex data often arise. Furthermore, this method can support the development of more flexible analytical tools for understanding the geometric properties of data in the context of probability and function analysis.

5. CONCLUSION

This work had looked at the geometric properties of generalized distribution associated with generalized Koebe distribution. It looked at the coefficient inequalities of the power series formed by its convolution and estimates the early coefficient bounds using Chebyschev polynomial. The results of this work can be used to model complex data by capturing no-normal, skewed or heavy tailed data. This study recommends utilizing the results in modeling complex data, especially those involving distributions with unique characteristics. In addition, it is suggested that further studies be conducted to explore other distributions or extend this method to practical applications in the field of statistics and data analysis. The development of more sophisticated numerical tools, such as the advanced use of Chebyshev polynomials, is also recommended to speed up the estimation of coefficients in real applications.

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